29 April 2025 **Review Class # 1**

HW10 Q3(a) Prove that

$$\sum_{1 \le n \le x} \frac{\sigma(n)}{n^2} = \frac{\pi^2}{6} \log x + O(1).$$

Observe that

$$\sum_{1 \le n \le x} \frac{\sigma(n)}{n^2} = \sum_{1 \le n \le x} \frac{1}{n^2} \sum_{d \mid n} d$$

or use $\frac{n}{d} \leftrightarrow d$

$$= \sum_{d \le x} \sum_{1 \le m \le x/d} \frac{1}{(md)^2} \cdot d$$

$(n = md)$

$$\boxed{\sigma(n) = \sum_{d \mid n} \varphi\left(\frac{n}{d}\right)\tau(d)}$$

$$= \sum_{d \le x} \frac{1}{d} \sum_{1 \le m \le x/d} \frac{1}{m^2}$$

Observe that

$$\sum_{1 \le m \le x/d} \frac{1}{m^2} = \underbrace{\sum_{m=1}^{\infty} \frac{1}{m^2}}_{\zeta(2) = \pi^2/6} + O\left(\frac{1}{x/d}\right)$$

Substitute:

$$\sum_{1\le n\le x}\frac{\sigma(n)}{n^2} = \sum_{d\le x}\frac{1}{d}\left(\frac{\pi^2}{6}+O\left(\frac{1}{x/d}\right)\right)$$

$$= \frac{\pi^2}{6}\sum_{d\le x}\frac{1}{d} + O\left(\sum_{d\le x}\frac{1}{x}\right)$$

$$= \frac{\pi^2}{6}\left(\log x + O(1)\right) + O(1)$$

$$= \frac{\pi^2}{6}\log x + O(1).$$

Q3 (b) Evaluate

$$\sum_{1\le n\le x} \frac{\varphi(n)}{n^2} = \sum_{1\le n\le x} \frac{1}{n^2}\cdot n\sum_{d\mid n}\frac{\mu(d)}{d}$$

using $\varphi(n) = n\sum_{d\mid n}\frac{\mu(d)}{d}$
$= n\prod_{p\mid n}\left(1-\frac{1}{p}\right)$

$$= \sum_{1\le d\le x}\ \sum_{1\le m\le x/d} \frac{1}{md}\cdot\frac{\mu(d)}{d}$$

$(n = md)$

$$= \sum_{1\le d\le x}\frac{\mu(d)}{d^2}\sum_{1\le m\le x/d}\frac{1}{m}$$

$$= \sum_{1\le d\le x}\frac{\mu(d)}{d^2}\left(\log\left(x/d\right) + O(1)\right)$$

$$= \sum_{1\le d\le x}\frac{\mu(d)}{d^2}\left(\log x - \log d + O(1)\right)$$

$$= (\log x)\sum_{1\le d\le x}\frac{\mu(d)}{d^2} + \underbrace{O\left(\sum_{1\le d\le x}\frac{|\mu(d)|}{d^2}\log(2d)\right)}_{=O(1)}$$

$$= (\log x)\left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\frac{1}{x}\right)\right) + O(1)$$

$$= (\log x) \cdot \frac{1}{\zeta(2)} + O(1)$$

$$= \frac{6}{\pi^2} \log x + O(1)$$

$$\sim$$

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right)}$$

$$\sum_{d=1}^{\infty} \prod_{p^h \| d} \frac{\mu(p^h)}{p^{2h}}$$

$$= \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{1}{\zeta(2)}.$$

Q5 Define

$$\alpha(n) = \prod_{\substack{1 \le a \le n \\ (a,n)=1}} a. \quad \text{Prove that } \alpha(n) = n^{\varphi(n)} \prod_{d \mid n} \left(\frac{d!}{d^d} \right)^{\mu(n/d)}.$$

Observe that

$$\log \alpha(n) = \sum_{\substack{1 \le a \le n \\ (a,n)=1}} \log a$$

If $(a,n) = k$, ~~then~~ $\displaystyle\sum_{d \mid k} \mu(d) = \begin{cases} 1, & \text{when } k=1 \\ 0, & \text{when } k>1. \end{cases}$

So
$$\log \alpha(n) = \sum_{1 \le a \le n} \left(\sum_{d \mid (a,n)} \mu(d) \right) (\log a)$$

$$= \sum_{d \mid n} \mu(d) \sum_{\substack{1 \le a \le n \\ d \mid a}} \log a.$$

$$= \sum_{d \mid n} \mu(d) \sum_{1 \le b \le n/d} \log(bd) \qquad a := bd$$

$$= \sum_{d \mid n} \mu(d) \sum_{1 \le b \le n/d} (\log b + \log d)$$

$$= \sum_{d \mid n} \mu(d) \left(\log\left(\left(\frac{n}{d}\right)!\right) + \frac{n}{d} \log d \right) \qquad d \leftrightarrow n/d$$

$$= \sum_{d \mid n} \mu(n/d) \left(\log(d!) + d \log(n/d) \right)$$

Q4 (b)

(i) State and ~~prove~~ the Möbius inversion formula.

If $f : \mathbb{N} \to \mathbb{C}$ and $g(n) := \sum_{d \mid n} f(d)$,

then $f(n) = \sum_{d \mid n} \mu(d)\, g(n/d)$

$$\left(= \sum_{d \mid n} \mu(n/d)\, g(d) \quad \checkmark \right)$$

(ii) Define the multiplicative function $g : \mathbb{N} \to \mathbb{Z}$ by putting, for each prime p,

$$g(p^h) = (h+1)^2 \qquad (h \geq 0).$$

Also, denote by $\mu(n)$ the Möbius function, and $\tau(n)$ – the number of positive divisors of n. Prove that

$$g(n) = \sum_{d \mid n} \tau(d^2) \quad \text{and} \quad \tau(n^2) = \sum_{d \mid n} g(d)\, \mu(n/d).$$

Observe that since $\tau(n)$ is a multiplicative function, so is $\tau(n^2)$. $\left((n,m)=1 \Rightarrow \tau((nm)^2) = \tau(n^2)\tau(m^2) \right)$

since $(n^2, m^2) = 1$

$$\Downarrow$$

$\sum_{d \mid n} \tau(d^2)$ is also multiplicative

Suffices to check that $g(n) = \sum_{d \mid n} \tau(d^2)$ holds whenever $n = p^h$, any prime p.

But

$$\sum_{d \mid p^h} \tau(d^2) = \sum_{\ell=0}^{h} \tau(p^{2\ell}) = \sum_{\ell=0}^{h} (2\ell+1)$$

$d = p^\ell$

$$= 2 \cdot \tfrac{1}{2} h(h+1) + (h+1)$$

$$= (h+1)^2. = g(p^h). \checkmark$$

By multiplicativity, have $g(n) = \sum_{d \mid n} \tau(d^2)$. $\square$

By Möbius inversion, we infer that

$$\tau(n^2) = \sum_{d \mid n} \mu(n/d)\, g(d)$$

$\square$ //

(c) Write

$$\tau^*(n) = \sum_{d^2 \mid n} 1 .$$

Prove that when X is large, one has

$$\sum_{1 \leq n \leq X} \tau^*(n) = \frac{\pi^2}{6} X + O(X^{1/2}).$$

Observe that

$$\sum_{1 \leq n \leq X} \tau^*(n) = \sum_{1 \leq n \leq X} \sum_{d^2 \mid n} 1$$

$$= \sum_{1 \leq d \leq X^{1/2}} \sum_{1 \leq m \leq X/d^2} 1 \qquad n = md^2$$

$$= \sum_{1 \le d \le x^{1/2}} \left(\frac{x}{d^2} + O(1) \right)$$

$$= x \sum_{1 \le d \le x^{1/2}} \frac{1}{d^2} + O\Big(\underbrace{\sum_{1 \le d \le x^{1/2}} 1}_{O(x^{1/2})} \Big)$$

$$= x \left(\sum_{d=1}^{\infty} \frac{1}{d^2} + O\Big(\underbrace{\sum_{d > x^{1/2}} \frac{1}{d^2}}_{O(1/x^{1/2})} \Big) \right) + O(x^{1/2})$$

$$= x \underbrace{\sum_{d=1}^{\infty} \frac{1}{d^2}}_{\zeta(2) = \frac{\pi^2}{6}} + O(x^{1/2})$$

$$= \frac{\pi^2}{6} x + O(x^{1/2}) . \quad //$$

HW7

Q2. Suppose $p > 3$ is prime.

(a) Find modulo p the sum, and the product, of all the distinct quadratic residues modulo p.

Let g be a primitive root mod p. Observe ~~that~~ the quadratic residues modulo p are given by

$$g^{2r} \quad \left(0 \le r < \tfrac{1}{2}(p-1)\right).$$

[Observe ~~that~~ $\left(\frac{g^{2r+1}}{p}\right) = \left(\frac{g}{p}\right) \equiv g^{\frac{1}{2}(p-1)} \pmod p$, and $g^{\frac{1}{2}(p-1)} \not\equiv 1 \pmod p$, so $g^{\frac{1}{2}(p-1)} \equiv -1 \pmod p$ so $\left(\frac{g^{2r+1}}{p}\right) = -1$.]

Then

$$\sum_{r=0}^{\frac{1}{2}(p-1)-1} g^{2r} = \frac{1-g^{2\cdot\frac{1}{2}(p-1)}}{1-g^2} = \frac{1-g^{p-1}}{1-g^2} \equiv 0 \pmod{p}$$

since $g^{p-1} \equiv 1 \pmod{p}$ (FLT)

and $g^2 \not\equiv 1 \pmod{p}$ since $p>3$.

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$$\prod_{r=0}^{\frac{1}{2}(p-1)-1} g^{2r} = g^{2\sum_{r=0}^{\frac{1}{2}(p-1)-1} 1} = g^{2\cdot\frac{1}{2}\cdot\frac{1}{2}(p-1)\cdot\frac{1}{2}(p-3)}$$

$$= \left(g^{\frac{1}{2}(p-1)}\right)^{\frac{1}{2}(p+1)} \equiv (-1)^{\frac{1}{2}(p-3)} \pmod{p}.$$

using $g^{\frac{1}{2}(p-1)} \equiv -1 \pmod{p}$. $\frac{1}{2}(p-3)$